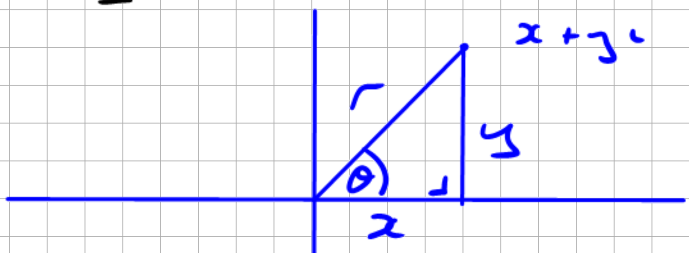
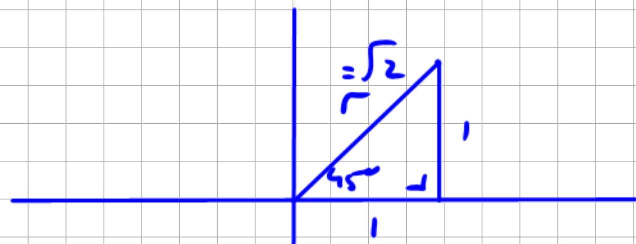
Polar Form



$z = x + yi$

$r = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x}$



$z = 1 + i$

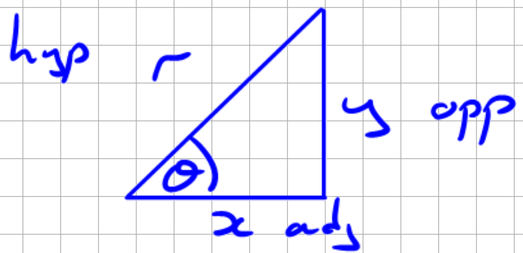
$r = \sqrt{2}$

$\tan\theta = 1$

$\theta = 45°$

$z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}(\cos 45 + i\sin 45)$



$z = x + yi$

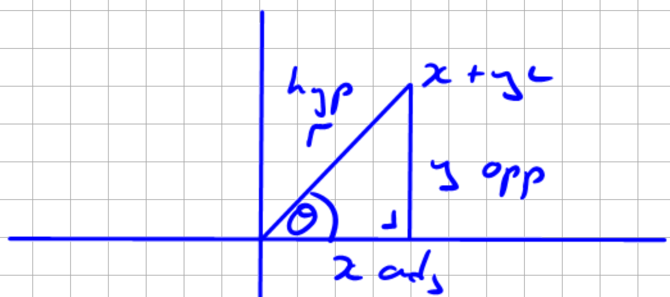
$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$z = r\cos\theta + i\,r\sin\theta$

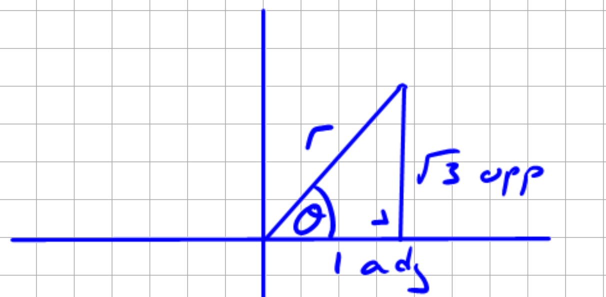
$z = r(\cos\theta + i\sin\theta)$ Polar Form

Write $z = 1 + \sqrt{3}i$ in Polar Form.



$r = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x}$



$r = \sqrt{1+3}$

$r = 2$

$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\theta = 60°$

$2(\cos 60 + i\sin 60)$

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

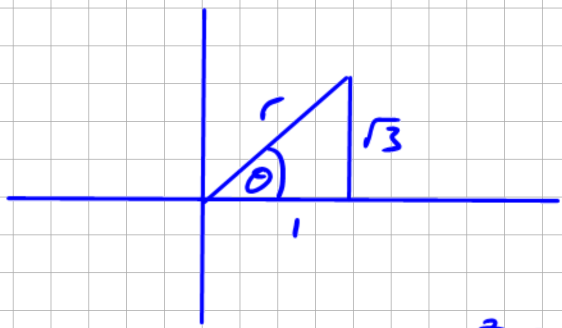
$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$$x + yi = r\cos\theta + i\,r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

Write $z = 1 + \sqrt{3}i$ and $w = -\sqrt{3} + i$ in polar form find (i) zw.

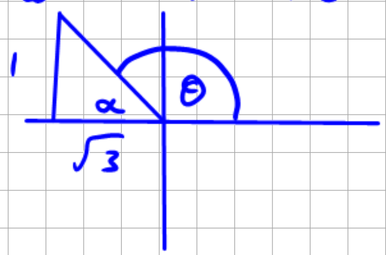
(ii) $\dfrac{w}{z}$

(iii) z^3



$$r = \sqrt{1+3} = 2$$
$$\tan\theta = \sqrt{3}$$
$$\theta = 60^\circ$$
$$z = r(\cos\theta + i\sin\theta)$$
$$= 2(\cos 60 + i\sin 60)$$

$w = -\sqrt{3} + i$ =



$$r = \sqrt{3+1} = 2$$
$$\tan\alpha = \frac{1}{\sqrt{3}}$$
$$\alpha = 30$$
$$\theta = 150^\circ$$
$$w = 2(\cos 150 + i\sin 150)$$

Properties of Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$$

$$\frac{1}{z_1} = \frac{1}{r_1}\left(\cos\theta_1 - i\sin\theta_1\right)$$

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right)$$

$$z_1^2 = z_1 z_1 = r_1^2 (\cos 2\theta_1 + i \sin 2\theta_1)$$

$$z_1^3 = z_1^2 . z_1 = r_1^3 (\cos 3\theta_1 + i \sin 3\theta_1)$$

De Moivre's Theorem.

$$z = r (\cos\theta + i \sin\theta)$$

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

$$z = 2(\cos 60 + i \sin 60)$$

$$w = 2(\cos 150 + i \sin 150)$$

$$zw = 4(\cos 210 + i \sin 210)$$

$$4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i \right) = -2\sqrt{3} - 2i$$

$$\frac{w}{z} = \frac{2}{2}\left(\cos(150 - 60) + i \sin(150 - 60) \right)$$

$$= \cos 90 + i \sin 90$$

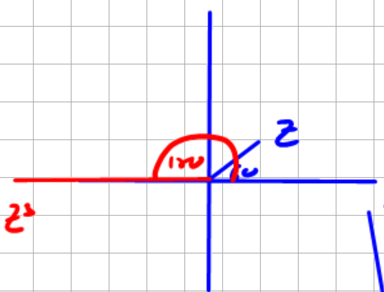
$$= 0 + i$$

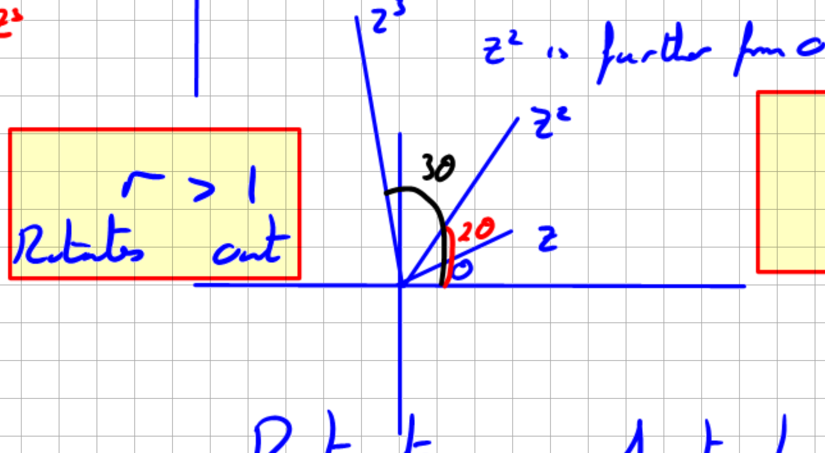
Rectangular form must put in 0.

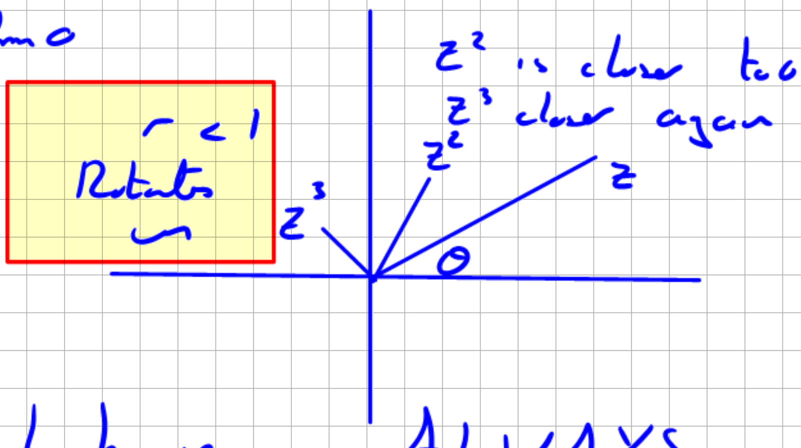
$$z^3 = \left(2(\cos 60 + i \sin 60) \right)^3$$

$$= 2^3 (\cos 180 + i \sin 180)$$

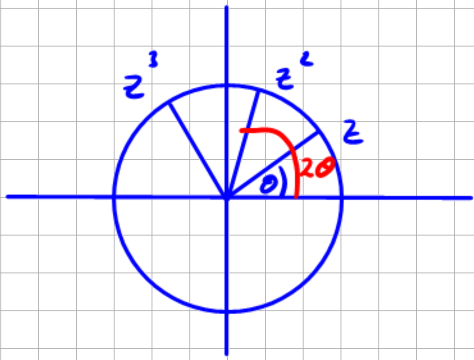
$$= 8(-1 + 0i) = -8 + 0i$$







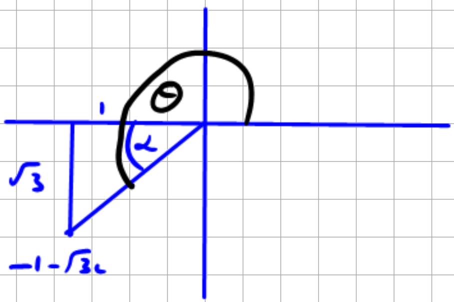
Rotate Anticlockwise ALWAYS



$r = 1$

There are 3 applications of De Moivre's Theorem.

Write $z = -1-\sqrt{3}i$ in Polar form hence find z^{10}.



$r = \sqrt{1+3} = 2$

$\tan \alpha = \sqrt{3}$

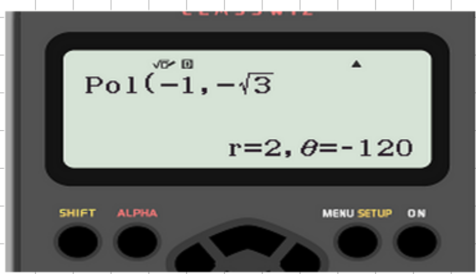
$\alpha = 60°$

$\theta = 240°$

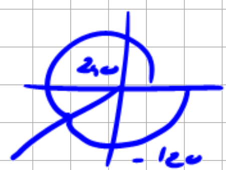
Shift Pol $\boxed{-1}$ Shift , $\boxed{-\sqrt{3}}$

Polar form.

)



-120



$360 - 120 = 240$

$-1-\sqrt{3}i = 2(\cos 240 + i\sin 240)$

$$(-1-\sqrt{3}i)^{10} = \left[2(\cos 240 + i\sin 240)\right]^{10}$$

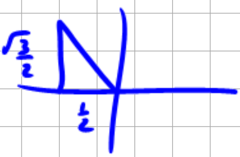
$$2^{10}(\cos 2400 + i\sin 2400)$$

$$2^{10}\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$$

$$2^{9}(-1-\sqrt{3}i) = -512 - 512\sqrt{3}i$$

$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Find z^2 and z^3

Show z, z^2 and z^3 on Argand diagram.

Rough 

Calculator

$r = 1 \qquad \theta = 120°$ (Positive)

$\theta =$ neg Ans $360 -$ cal

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos 120 + i\sin 120$$

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \left(\cos 120 + i\sin 120\right)^2$$

$$= \cos 240 + i\sin 240$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

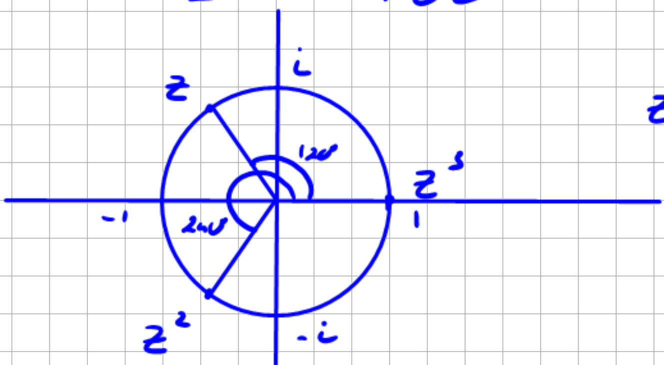
$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = \left(\cos 120 + i\sin 120\right)^3$$

$$= \cos 360 + i\sin 360$$

$$= 1 + 0i$$

r for each $= 1$

$\Rightarrow$ on a circle of radius 1.



$z^3 = (\cos\theta + i\sin\theta)^3$

$= \cos 3\theta + i\sin 3\theta$

$3\theta = 360$

$\theta = 120°$.

in this question $z^3 = \cos 360 + i\sin 360$

so $3\theta = 360°$.

Application 2. - find roots.

$z^4 = -8 + 8\sqrt{3}i$ find 4 values for z.

Not asked to find $(-8+8\sqrt{3}i)^4$.

Method above.

What are we being asked?

$$z^4 = -8 + 8\sqrt{3}i$$

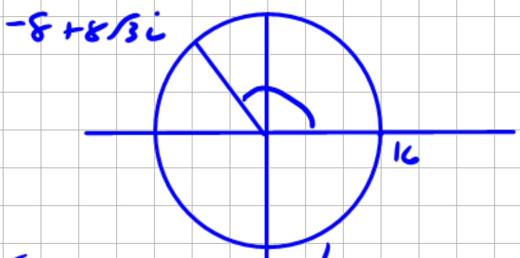
$$z = \sqrt[4]{(-8+8\sqrt{3}i)}$$

$$= (-8+8\sqrt{3}i)^{\frac{1}{4}}$$

Once there is fraction in power must use general polar form.

$-8+8\sqrt{3}i \Rightarrow$ cal $r = 16$ $\theta = 120$

$$-8+8\sqrt{3}i = 16\left(\cos(120+360n) + i\sin(120+360n)\right)$$



Angle $30 \times 4 = 120$

$120 \times 4 = 480 - 360 = 120$

$$(-8+8\sqrt{3}i)^{\frac{1}{4}} = 16^{\frac{1}{4}}\left(\cos\tfrac{1}{4}(120+\underline{360n}) + i\sin\tfrac{1}{4}(120+360n)\right)$$

$$= 2\left(\cos(30+90n) + i\sin(30+90n)\right)$$

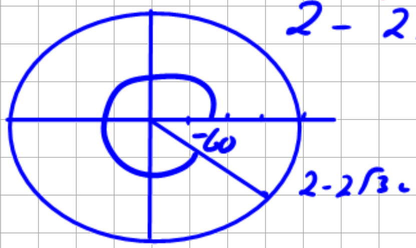
$n=0$ $= 2(\cos 30 + i\sin 30) = \sqrt{3} + i$

$n=1$ $= 2(\cos 120 + i\sin 120) = -1 + \sqrt{3}i$

$n=2$ $= 2(\cos 210 + i\sin 210) = -\sqrt{3} - i$

$n=3$ $= 2(\cos 300 + i\sin 300) = 1 - \sqrt{3}i$

Write $2-2\sqrt{3}i$ in general polar form. $z^2 = 2-2\sqrt{3}i$ find 2 values for z.



$2-2\sqrt{3}i$ $\quad r = 4 \quad \theta = 300$

$\Rightarrow$ Calculator $\uparrow$ $(-60) = 360-60$

$= 300$

$$2-2\sqrt{3}i = 4\left(\cos(300+360n) + i\sin(300+360n)\right)$$

Note $\underset{n=0}{\cos 120} = \underset{n=1}{\cos 480} = \underset{n=2}{\cos 840} =$ goes on $=$ general polar form.

$$z^2 = 2-2\sqrt{3}i \Rightarrow z = \left(2-2\sqrt{3}i\right)^{\frac{1}{2}}$$

<u>NB</u> When power is a fraction must use general polar form.

$$\left(2-2\sqrt{3}i\right)^{\frac{1}{2}} = \left[4\left(\cos(300+360n) + i\sin(300+360n)\right)\right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(\cos\tfrac{1}{2}(300+360n) + i\sin\tfrac{1}{2}(300+360n)\right)$$

$$= 2\left(\cos(150+180n) + i\sin(150+180n)\right)$$

$n=0 \quad 2(\cos 150 + i\sin 150)$

$$2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$n=1 \quad \left[2(\cos 330 + i\sin 330)\right]^2$

$$2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

$\cos 300 = \cos 660 =$ so on

Work backwards

$$\left[2(\cos 150 + i\sin 150)\right]^2 = 2^2(\cos 300 + i\sin 300)$$

$$\left[2(\cos 330 + i\sin 330)\right]^2 = 2^2(\cos 660 + i\sin 660)$$

Same

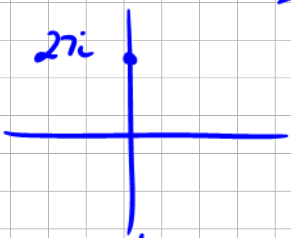
$\cos 300 = \cos 660 =$ revolution.

$z^3 = 27i$ find 3 values for z.

Said $(27i)^3$ = polar form

$z^3 = 27i$

$\Rightarrow \quad z = 27i^{\frac{1}{3}}$ = general polar form.



$r = 27 \qquad \theta = 90°$

$27i = 27(\cos(90 + 360n) + i\sin(90+360n))$

$z^{\frac{1}{3}} = 27^{\frac{1}{3}}\left(\cos\tfrac{1}{3}(90 + 360n) + i\sin\tfrac{1}{3}(90+360n)\right)$

$= 3(\cos(30 + 120n) + i\sin(30 + 120n))$

$n = 0 \quad = 3(\cos 30 + i\sin 30)$

$= 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$n = 1 \quad = 3(\cos 150 + i\sin 150)$

$= 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$n = 2 \quad = 3(\cos 270 + i\sin 270)$

$= 3(0 - i) = 0 - 3i$

$z^3 = 1$ find the 3 values for z.

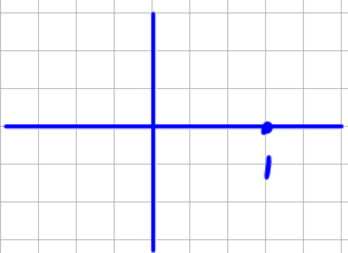
$z = 1^{\frac{1}{3}}$

$x^3 - 1 = 0$

$x^3 = 1$

$x = \sqrt[3]{1} = 1$

One real and 2 imaginary.



$r = 1 \qquad \theta = 0$

$1 = \cos 2n\pi + i\sin 2n\pi$

$1^{\frac{1}{3}} = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$

$360n = 2n\pi$

Degrees = radians

$n = 0 \qquad 1^{\frac{1}{3}} = 1$

$n = 1 \qquad 1^{\frac{1}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

EKKER

$= \cos 120 + i \sin 120$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$n = 2 \qquad 1^{\frac{1}{3}} = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

$= \cos 240 + i \sin 240$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$

$z^3 = 1$ then $z = 1 \qquad = 1$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i = \omega$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2} i = \omega^2$

Cube roots of unity.

Prove (i) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

(ii) $1 + \omega + \omega^2 = 0$

$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2 = \frac{1}{4} - \cancel{2}\left(\frac{1}{\cancel{2}} \cdot \frac{\sqrt{3}}{2} i\right) + \frac{3}{4} i^2$

$= \frac{1}{4} - \frac{\sqrt{3}}{2} i - \frac{3}{4} \qquad i^2 = -1$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

(ii) $1 - \frac{1}{2} + \cancel{\frac{\sqrt{3}}{2} i} - \frac{1}{2} - \cancel{\frac{\sqrt{3}}{2}} i = 0$

Application 3.

Use De Moivre's Theorem to prove
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Learn where to start.
Take $\cos\theta + i\sin\theta$ and cube it
in 2 different ways
(i) De Moivre's
(ii) Multiply out (Pascal's Triangle)

(i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

(ii) $(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$\underset{R}{\cos 3\theta} + i\sin 3\theta = \underset{R}{\cos^3\theta} + 3i\cos^2\theta\sin\theta - \underset{R}{3\cos\theta\sin^2\theta} - i\sin^3\theta$

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

Note: $\sin^2\theta + \cos^2\theta = 1$ = Tables.

$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$

Pascal's Triangle

$(a+b)^3$

$a^3 + 3a^2b + 3ab^2 + b^3$

1

1 1

1 2 1

1 3 3 1