2019.M29

2019L003A1EL

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination 2019

Mathematics

Paper 1

Higher Level

Friday 7 June – Afternoon 2:00 – 4:30

300 marks

Examination Number					

Centre Stamp

2019L003A1EL0128

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Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer all nine questions.

Write your Examination Number in the box on the front cover.

Write your answers in blue or black pen. You may use pencil in graphs and diagrams only.

This examination booklet will be scanned and your work will be presented to an examiner on screen. Anything that you write outside of the answer areas may not be seen by the examiner.

Write all answers into this booklet. There is space for extra work at the back of the booklet.
If you need to use it, label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

You will lose marks if your solutions do not include relevant supporting work.

You may lose marks if the appropriate units of measurement are not included, where relevant.

You may lose marks if your answers are not given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A	Concepts and Skills	150 marks

Answer **all six** questions from this section.

Question 1 **(25 marks)**

(a) In the expansion of $(2x+1)(x^2+px+4)$, where $p \in \mathbb{N}$, the coefficient of x is twice the coefficient of x^2. Find the value of p.

$(2x+1)(x^2+px+4)$

$2x^3+2px^2+8x+x^2+px+4$

$x^2 = 2p+1$

$x = 8+p$

$8+p = 2(2p+1)$

$8+p = 4p+2$

$6 = 3p \Rightarrow p = 2$

(b) Solve the equation $\frac{3}{2x+1} + \frac{2}{5} = \frac{2}{3x-1}$ where $x \neq -\frac{1}{2}, \frac{1}{3}$, and $x \in \mathbb{R}$.

$$\frac{3(5)(3x-1) + 2(2x+1)(3x-1) = 2(5)(2x+1)}{5(2x+1)(3x-1)}$$

$$15(3x-1) + 2(6x^2 - 2x + 3x - 1) = 10(2x+1)$$

$$45x - 15 + 12x^2 + 2x - 2 = 20x + 10$$

$$12x^2 + 27x - 27 = 0$$

$$4x^2 + 9x - 9 = 0 \qquad \text{CN } -36$$

$$4x^2 + 12x - 3x - 9 = 0 \qquad \text{Sub } 9$$

$$4x(x+3) - 3(x+3) = 0 \qquad 1 \times 36$$

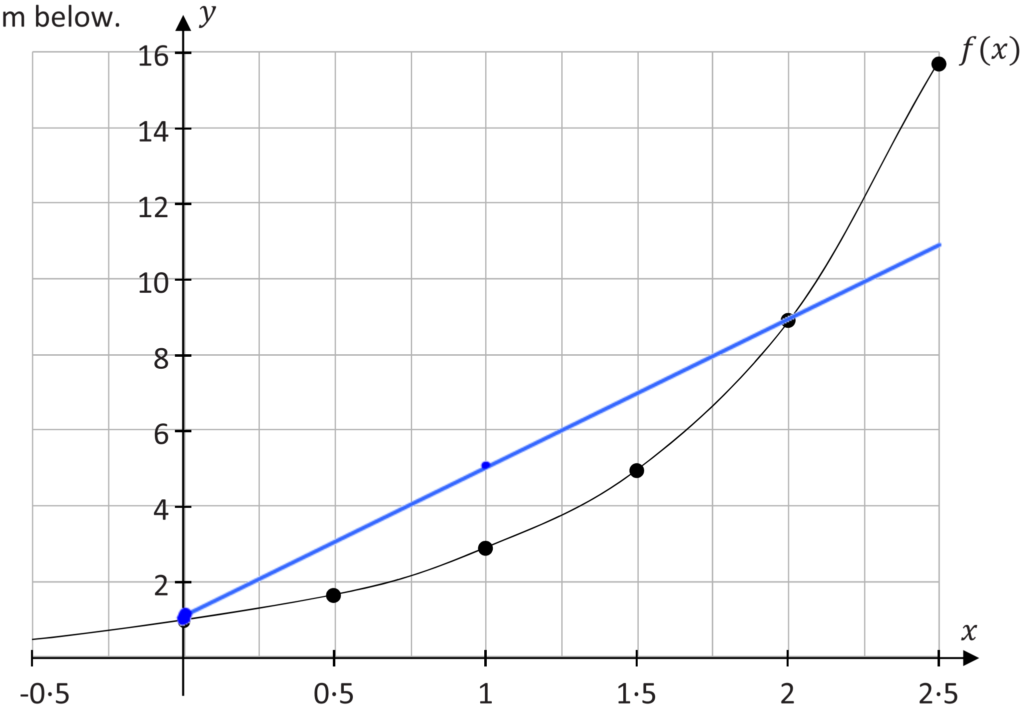
$$2 \times 18$$

$$(x+3)(4x-3) = 0 \qquad 3 \times 12$$

$$x = -3 \qquad x = \frac{3}{4}$$

Question 2 **(25 marks)**

The graph of the function $f(x) = 3^x$, where $x \in \mathbb{R}$, cuts the y-axis at $(0, 1)$ as shown in the diagram below.



(a) **(i)** Draw the graph of the function $g(x) = 4x + 1$ on the diagram.

$g(0) = 1 \quad (0, 1)$

$g(1) = 4(1) + 1 = 5 \quad (1, 5)$ ✓

(ii) Use substitution to verify that $f(x) < g(x)$, for $x = 1{\cdot}9$.

$f(1.9) < g(1.9)$

$3^{1.9} < 4(1.9) + 1$

$8.0 \leq 8.6$ ✓

(b) Prove, using induction, that $f(n) \geq g(n)$, where $n \geq 2$ and $n \in \mathbb{N}$.

Step 1 Prove for $n = 2$

$3^2 \geq 4(2) + 1$

$9 \geq 9$ ✓

Step 2: Assume $n = k$ $3^k \geq 4k + 1$

Step 3: Prove $n = k + 1$

$3^{k+1} \geq 4(k+1) + 1$

$3(3^k) \geq 4k + 5$

$3(4k+1) \geq 4k + 5$

$12k + 3 \geq 4k + 5$ $k \geq 2$

$3(3^k) \geq 12k + 3 \geq 4k + 5$

$3(3^k) \geq 4k + 5$

Since true for $n = 2$ and $n = k + 1$ then true for all $n \geq 2$.

Question 3 **(25 marks)**

(a) Factorise fully: $3xy - 9x + 4y - 12$.

$3x(y-3)+4(y-3)$

$(y-3)(3x+4)$

(b) $g(x) = 3x\ln x - 9x + 4\ln x - 12$.
Using your answer to **part (a)** or otherwise, solve $g(x) = 0$.

$(\ln x - 3)(3x+4) = 0$

$\ln x = 3$

$\log_e x = 3$

$e^3 = x$

$\log_b n = p$

$b^p = n$

$\cancel{3x + 4 = 0}$

$\cancel{3x = -4}$

(c) Evaluate $g'(e)$ correct to 2 decimal places.

$$g(x) = 3x\ln x - 9x + 4\ln x - 12$$

$$g'(x) = 3\cancel{x}\cdot\frac{1}{\cancel{x}} + 3\ln x - 9 + 4\cdot\frac{1}{x}$$

$$g'(x) = 3\ln x - 6 + \frac{4}{x}$$

$$g'(e) = 3\ln e - 6 + \frac{4}{e}$$

$\ln e = 1$

$$= \frac{4}{e} - 3$$

$$= -1.528 = -1.53$$

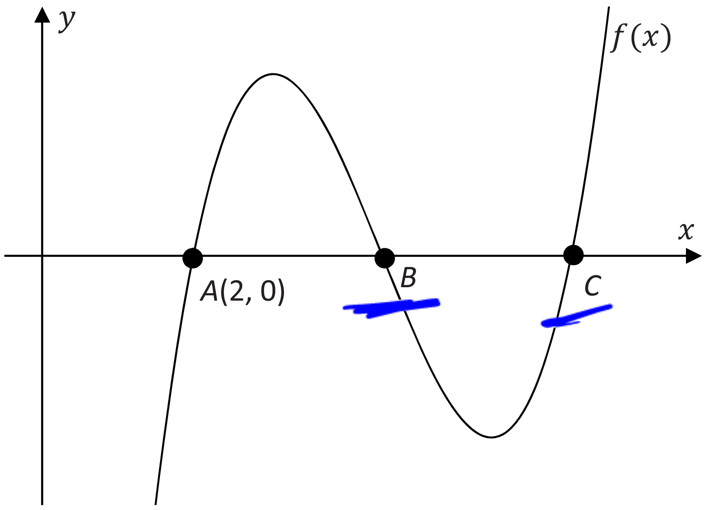
Question 4 **(25 marks)**

(a) Find $\int(4x^3 - 6x + 10)\,dx$.

$\frac{4x^4}{4} - \frac{6x^2}{2} + 10x + c$

$x^4 - 3x^2 + 10x + c$

(b) Part of the graph of a cubic function $f(x)$ is shown below (graph not to scale). The graph cuts the x-axis at the three points $A(2, 0)$, B, and C.



(i) Given that $f'(x) = 6x^2 - 54x + 109$, show that $f(x) = 2x^3 - 27x^2 + 109x - 126$.

$\int(6x^2 - 54x + 109)\,dx$

$f(x) = \frac{6x^3}{3} - \frac{54x^2}{2} + 109x + c$

$f(x) = 2x^3 - 27x^2 + 109x + c$

(2.0)

$f(2) = 2(2)^3 - 27(2)^2 + 109(2) + c = 0$

$126 + c = 0$

$c = -126$

(ii) Find the co-ordinates of the point *B* and the point *C*.

$(2,0)$ $x=2$ is a root

$x-2$ is a factor

$$\begin{array}{r} 2x^2-23x+63 \\ x-2\,\overline{)\,2x^3-27x^2+109x-126} \\ -(2x^3-4x^2) \\ -23x^2+109x \\ -(-23x^2+46x) \\ 63x-126 \\ 63x-126 \\ 0 \end{array}$$

$2x^2-23x+63=0$

$(2x-9)(x-7)=0$

$x=\frac{9}{2}$ $x=7$

$B = \left(\frac{9}{2}, 0\right)$ $C = (7, 0)$

Question 5 **(25 marks)**

(a) $3+2i$ is a root of $z^2+pz+q=0$, where $p, q \in \mathbb{R}$, and $i^2=-1$.
Find the value of p and the value of q.

$3+2i$ other $3-2i$

$p = -(3+2i+3-2i)$

$= -6$

$q = (3+2i)(3-2i) = 9-4i^2$

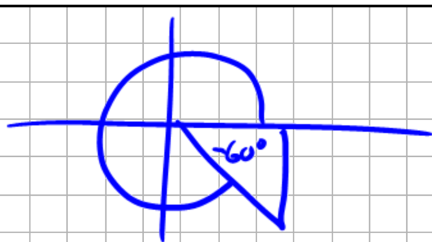
$= 13$

$(3+2i)^2 + p(3+2i) + q = 0 + 0i$

(b) **(i)** $v = 2-2\sqrt{3}i$. Write v in the form $r(\cos\theta + i\sin\theta)$,
where $r \in \mathbb{R}$ and $0 \le \theta \le 2\pi$.

$r = 4 \quad \theta = -60$

$r = 4 \quad \theta = 300$



$180° = \pi$

$1° = \frac{\pi}{180}$

$300° = \frac{300\pi}{180} = \frac{5\pi}{3}$

$4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

(ii) Use your answer to **part (b)(i)** to find the **two** possible values of w, where $w^2 = v$.
Give your answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

$$w^2 = 4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

$$w = \left[4\left(\cos\left(\frac{5\pi}{3} + 2n\pi\right) + i\sin\left(\frac{5\pi}{3} + 2n\pi\right)\right)\right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(\cos\tfrac{1}{2}(300 + 360n) + i\sin\tfrac{1}{2}(300 + 360n)\right)$$

$$= 2\left(\cos(150 + 180n) + i\sin(150 + 180n)\right)$$

$n = 0$

$$2(\cos 150 + i\sin 150)$$

$$2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$n = 1$

$$2(\cos 330 + i\sin 330)$$

$$2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

Question 6 **(25 marks)**

(a) **(i)** Given that $x - \sqrt{32} = \sqrt{128} - 5x$, find the value of x, where $x \in \mathbb{R}$.
Give your answer in the form $a\sqrt{2}$, where $a \in \mathbb{N}$.

$6x = \sqrt{128} + \sqrt{32}$

$6x = 12\sqrt{2}$

$x = 2\sqrt{2}$

(ii) $A = \left\{\sqrt{32k^2}, \sqrt{50k^2}, \sqrt{128k^2}, \sqrt{98k^2}\right\}$, where $k \in \mathbb{N}$.

Show that the **mean** of set A is equal to the **median** of set A.

Mean	Median
$\dfrac{4+5+7+8}{4}$	$\sqrt{ab} = \sqrt{a}\sqrt{b}$
$\dfrac{24}{4}$	$\sqrt{k^2} = k$
$6\sqrt{2}\,k$	$\sqrt{32k^2}, \sqrt{32}\,k$
	$4\sqrt{2}k, 5\sqrt{2}k$
	$\sqrt{64}\sqrt{2} = 8\sqrt{2}k \quad 7\sqrt{2}k$
	$4\sqrt{2}k, 5\sqrt{2}k, 7\sqrt{2}k, 8\sqrt{2}k$
	$6\sqrt{2}k$

(b) Prove, using contradiction, that $\sqrt{2}$ is **not** a rational number.

$\sqrt{2}$ is a rational

$\sqrt{2} = \frac{p}{q}$ where p, q have no common factor.

$2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2$

$\Rightarrow p^2$ must be even

$\sqrt{16} = 4$

$\sqrt{36} = 6$

$\sqrt{25} = 5$

p must be even

$p = 2k \Rightarrow p^2 = 4k^2$

$4k^2 = 2q^2$

$q^2 = 2k^2 \Rightarrow q^2$ is even

q is even

p, q have common factor 2.

$\Rightarrow$ contrac

$\Rightarrow \sqrt{2}$ is irrational.

Section B — Contexts and Applications — 150 marks

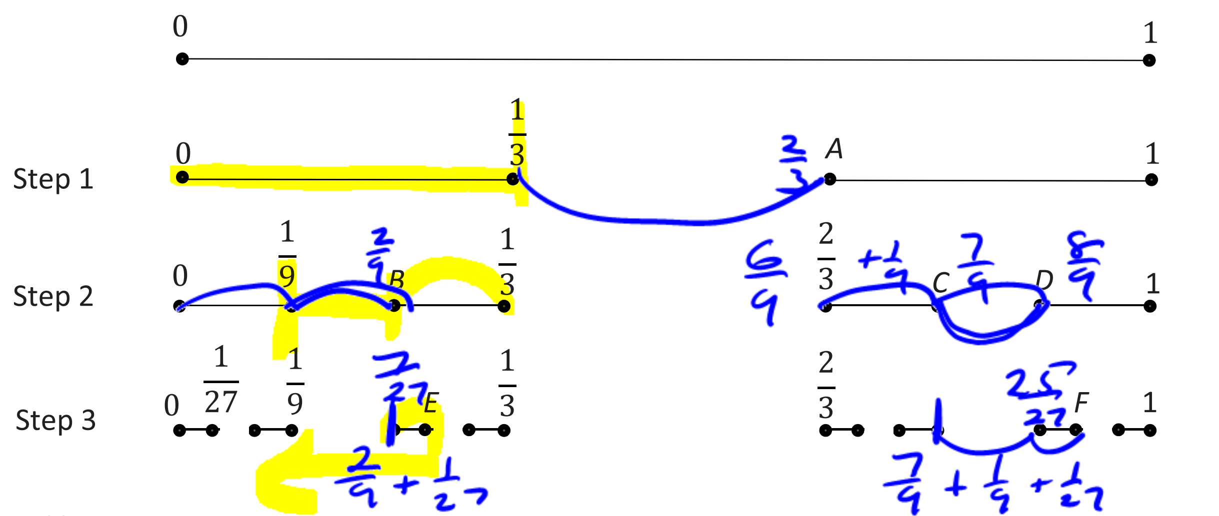
Answer **all three** questions from this section.

Question 7 **(45 marks)**

The closed line segment [0, 1] is shown below. The first three steps in the construction of the *Cantor Set* are also shown:

- Step 1 removes the open middle third of the line segment [0, 1] leaving two **closed line segments** (i.e. the end points of the segments remain in the *Cantor Set*)
- Step 2 removes the middle third of the two remaining segments leaving four closed line segments
- Step 3 removes the middle third of the four remaining segments leaving eight closed line segments.

The process continues **indefinitely**. The set of points in the line segment [0, 1] that are **not** removed during the process is the *Cantor Set.*



(a) **(i)** Complete the table below to show the length of the line segment(s) removed **at each step** for the first 5 steps. Give your answers as fractions.

Step	Step1	Step 2	Step 3	Step 4	Step 5
Length Removed	$\frac{1}{3}$ ✓	$\frac{2}{9}$ ✓	$\frac{4}{27}$	$\frac{8}{81}$	$\frac{16}{243}$

$$\frac{1}{3} \times \frac{2}{3} = \frac{2}{9} \times \frac{2}{3} = \frac{4}{27} \times \frac{2}{3}$$

(ii) Find the **total** length of all of the line segments removed from the initial line segment of length 1 unit, after a finite number (n) of steps in the process.
Give your answer in terms of n.

$\frac{1}{3}, \frac{2}{9}, \frac{4}{27}$ $\quad a = \frac{1}{3}$ $\quad r = \frac{2}{3}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{\cancel{\frac{1}{3}}\left(1-\left(\frac{2}{3}\right)^n\right)}{\cancel{\frac{1}{3}}} = 1-\left(\frac{2}{3}\right)^n$$

(iii) Find the total length removed, from the initial line segment, after an infinite number of steps of the process.

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$$

(b) (i) Complete the table below to identify the end-points labelled in the diagram.
Give your answers as **fractions.**

Label	A	B	C	D	E	F
End-point	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{7}{27}$	$\frac{25}{27}$

This question continues on the next page.

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(ii) Give a reason why $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81}$ is a point in the *Cantor Set*.

End Pt.

(iii) The limit of the series $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \cdots$ is a point in the *Cantor Set*. Find this point.

$a = \frac{1}{3} \quad r = -\frac{1}{3}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4}$

Question 8 **(50 marks)**

The weekly revenue produced by a company manufacturing air conditioning units is seasonal.
The revenue (in euro) can be approximated by the function:

$$r(t) = 22\,500\cos\left(\frac{\pi}{26}t\right) + 37\,500, \qquad t \geq 0$$

where t is the number of weeks measured from the beginning of July and $\left(\frac{\pi}{26}t\right)$ is in radians.

(a) Find the approximate revenue produced 20 weeks after the beginning of July.
Give your answer correct to the nearest euro.

$t = 20$

$r(20) = 22500\cos\left(\frac{20\pi}{26}\right) + 37500$

$= 20658.51$

$= €20,659$

(b) Find the two values of the time t, within the first 52 weeks, when the revenue is approximately €26 250.

$r(t) = 26250$

$22500\cos\left(\frac{\pi}{26}t\right) + 37500 = 26250$

$22500\cos\left(\frac{\pi}{26}t\right) = -11250$

$\cos\left(\frac{\pi}{26}t\right) = -\frac{1}{2}$

$\cos\left(\frac{\pi}{26}t\right) = \frac{1}{2} \Rightarrow \frac{\pi}{26}t = \frac{\pi}{3}$

$\pi - \theta$ θ

S A

T C

$\pi + \theta$ $2\pi - \theta$

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\frac{6\pi}{3} - \frac{\pi}{3} = \frac{5\pi}{3}$

$\frac{\pi}{26}t = \frac{2\pi}{3}$

$t = \frac{52}{3}$

$\frac{\pi}{26}t = \frac{5\pi}{3}$

$t = \frac{103}{3}$

This question continues on the next page.

(c) Find $r'(t)$, the derivative of $r(t) = 22\,500\cos\left(\frac{\pi}{26}t\right) + \cancel{37\,500}$.

$$r'(t) = 22500\left(\frac{\pi}{26}\right)\left(-\sin\frac{\pi}{26}t\right)$$

$$= \frac{-11250}{13}\pi\sin\left(\frac{\pi}{26}t\right)$$

(d) Use calculus to show that the revenue is increasing 30 weeks after the beginning of July.

$t = 30$

$$\frac{-11250}{13}\pi\sin\left(\frac{\pi(30)}{26}\right)$$

$402\pi > 0 \Rightarrow$

Increasing

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(e) Find a value for the time t, within the first 52 weeks, when the revenue is at a minimum. Use $r''(t)$, to verify your answer.

$$-\frac{11252}{13}\pi \sin\left(\frac{\pi}{26}t\right) = 0$$

$$\sin \frac{\pi}{26}t = 0$$

$$\frac{\pi}{26}t = 0 \qquad \frac{\pi}{26}t = \pi$$

$$t = 0 \qquad t = 26$$

$$r'(t) = -\frac{11252}{13}\pi \sin\left(\frac{\pi}{26}t\right)$$

$$r''(t) = -\frac{11252}{13}\pi \cdot \frac{\pi}{26} \cos\left(\frac{\pi}{26}t\right)$$

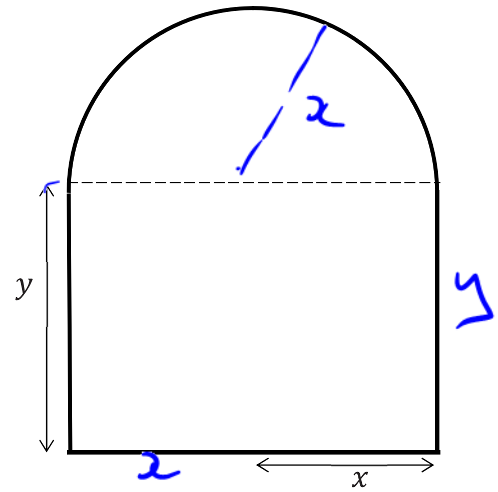
$$r''(0) = < 0 \Rightarrow \text{Maximum}$$

$$r''(26) > 0 \Rightarrow \text{Minimum.}$$

$$t = 26.$$

Question 9 **(55 marks)**

2



Photograph by Lionel Wall.
http://greatenglishchurches.co.uk/html/castle_rising/html

Norman windows consist of a rectangle topped by a semi-circle as shown above.
Let the height **of the rectangle** be y metres and the radius of the semi-circle be x metres as shown. The perimeter of the window is *P*.

(a) (i) Write *P* in terms of x, y, and π.

$l = 2\pi r = 2\pi x$

$P = 2x + 2y + \pi x$

(ii) In a particular Norman window the perimeter *P* = 12 metres.
Show that $y = \dfrac{12-(2+\pi)x}{2}$ for $0 \leq x \leq \frac{12}{2+\pi}$ where $x \in \mathbb{R}$.

$2x + 2y + \pi x = 12$

$2y = 12 - 2x - \pi x$

$y = \dfrac{12 - (2+\pi)x}{2}$

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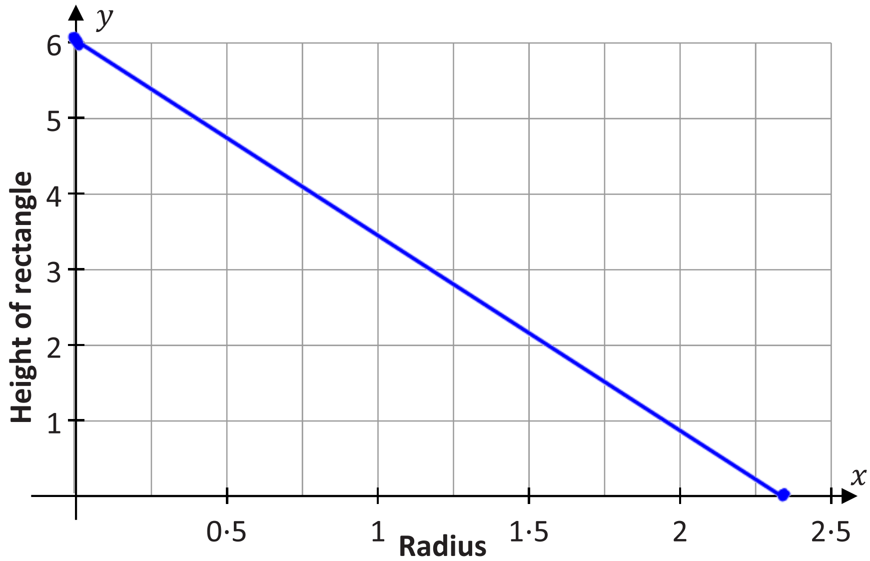
(b) **(i)** Complete the table on the right.

$y = \dfrac{12 - \cancel{(2+\pi)} \dfrac{12}{\cancel{2+\pi}}}{2}$

x	0	$\dfrac{12}{2+\pi}$
$y = \dfrac{12-(2+\pi)x}{2}$	6	0

(ii) On the diagram below, draw the graph of the linear function, $y = \frac{12-(2+\pi)x}{2}$ for $0 \leq x \leq \frac{12}{2+\pi}$ where $x \in \mathbb{R}$.



(iii) Find the slope of the graph of y, correct to 2 decimal places.
Interpretation this slope in the context of the question.

Slope:

$y = \dfrac{12-(2+\pi)x}{2} = 6 - \left(\dfrac{2+\pi}{2}\right)x$

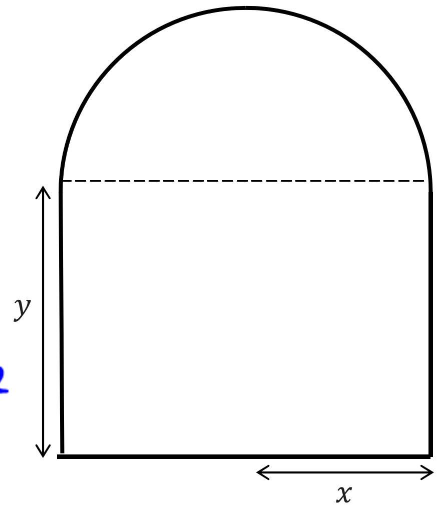
$m = -\dfrac{2+\pi}{2} = -2.57$

Interpretation: Radius increases by 1 the height is reduced by 2.57.

This question continues on the next page.

(c) **(i)** The Norman window shown below has a perimeter of 12 metres and $y = \frac{12-(2+\pi)x}{2}$.

Show that the function $a(x) = \frac{24x-(\pi+4)x^2}{2}$ represents the area of the window, in terms of x and π.



$$a = 2xy + \frac{\pi x^2}{2}$$

$$= 2x\left[\frac{12-(2+\pi)x}{2}\right] + \frac{\pi x^2}{2}$$

$$= \frac{24x - 4x^2 - 2\pi x^2 + \pi x^2}{2}$$

$$= \frac{24x - (\pi+4)x^2}{2}$$

(ii) Find $a'(x)$.

$$a'(x) = \frac{24 - 2x(\pi+4)}{2}$$

$$= 12 - x(\pi+4)$$

(iii) Find the relationship between x and y when the area of the window in **part (c)(i)** is at its maximum.

$$12 - (\pi + 4)x = 0$$

$$(\pi + 4)x = 12$$

$$x = \frac{12}{\pi + 4}$$

$$y = \frac{12 - (2 + \pi)x}{2}$$

$$= \frac{12 - (2 + \pi)\cdot\frac{12}{\pi + 4}}{2}$$

$$= \frac{\frac{12(\pi + 4) - 12(2 + \pi)}{\pi + 4}}{\frac{2}{1}}$$

$$= \frac{\cancel{12\pi} + 48 - 24 - \cancel{12\pi}}{2(\pi + 4)}$$

$$\frac{12}{\pi + 4} = x$$

$$x = y.$$

You may use this page for extra work.
Label any extra work clearly with the question number and part.

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Leaving Certificate 2019 – Higher Level

Mathematics – Paper 1

Friday 7 June
Afternoon 2:00 to 4:30

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